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# The use of a variational procedure in studying coupled dielectric waveguides

**P D Hollister and Raymund C Jones**

The School of Physics and Astronomy, The University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

E-mail: pdh@th.ph.bham.ac.uk

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**Abstract**

We consider the propagation of electromagnetic waves in an optical device known as a 'directional coupler', which is widely used in telecommunication systems. This device consists of a pair of dielectric waveguides in close proximity. Such waveguides are strongly coupled and there is energy transfer between the waveguides.

The propagation of light in such waveguides is *paraxial* and can be described by a parabolic differential equation closely resembling the Schrödinger equation for motion of a quantum particle in a two-dimensional time-dependent potential well. We exploit the quantum mechanical analogue of the optical system to write the *propagator* describing paraxial propagation as a path integral over optical paths.

We use the Feynman–Kleinert variational procedure to calculate approximate expressions for the propagation constant and the field profile of the lowest-order mode of the waveguiding system. An approximate expression for the beat length of the system is also calculated in the case where the waveguides are strongly coupled. The results are found to be in better agreement with other theoretical calculations than are the results of a previous variational calculation.

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## 1. Introduction

Fibre-optic and integrated optical devices are widely used in telecommunication systems and consist of a large number of interconnected dielectric waveguide sections. An important waveguide structure in such a device is a pair of dielectric waveguides in close proximity forming what is known as a 'directional coupler'. A simple directional coupler consisting of parallel waveguide sections is illustrated in figure 1. Such waveguides may be strongly coupled

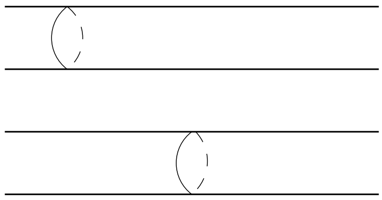

**Figure 1.** A directional coupler consisting of a pair of parallel waveguide sections is a typical waveguide structure encountered in integrated optics.

so that there is energy transfer between the waveguides [1]. The close proximity of the two parallel waveguides illustrated in figure 1 produces strong coupling, the effect of which is to produce a periodic variation of intensity along each guide. This periodic intensity variation is usually thought of as a beat phenomenon, and the distance measured in the direction of propagation over which a period of this variation occurs is known as the beat length of the system [2]. A complete description of the propagation in such a structure should include a calculation of the propagation constants and field profiles of the modes that are supported by the system.

The exchange of power between two waveguides in close proximity has useful applications relating to modulators, switches and signal processing [3]. The importance of a waveguide structure similar to that illustrated in figure 1 has led to a large number of theoretical techniques being applied to the study of such guiding systems.

A numerical technique that is widely used to study the propagation of light in the complicated waveguide structures that are encountered in integrated optics is the *beam propagation method* [4, 5] in which the propagating fields are decomposed into guided modes and the propagation of each mode is considered by solving an approximate version of the wave equation. The beam propagation method has been applied to the study of integrated optical coupling structures, including directional couplers, with a large degree of success [5, 6]. The calculations in [6] are only applicable in the adiabatic limit, in which the refractive index of the waveguiding system varies slowly with position on the scale of a typical wavelength of the propagating light. The advanced beam propagation method presented in [5] overcomes this limitation by including a full description of the coupling between the transverse and axial components of the propagating fields.

Another theoretical technique which has been applied to the study of propagation in parallel waveguide structures is the so-called *coupled mode theory* [2, 7]. This latter theory develops a set of coupled wave equations which relate the fields of the guided modes of one fibre to the fields of the guided modes of another fibre, which is coupled to the first because of its close proximity. By considering the propagation of single modes in each fibre it is possible to obtain expressions describing the periodic exchange of energy between two parallel guides [7]. Expressions for the field profiles and propagation constants of the modes supported by the system can also be obtained using this theory [2, 8].

A number of assumptions are conventionally made in the study of systems via the coupled mode theory, but it has been shown by Hardy and Streifer [3] that these assumptions are not always valid. The results of conventional coupled mode theory are only accurate when the guides are nearly identical and are weakly coupled. This is due to the fact that the conventional theory ignores the overlap between the guided modes in the two fibres [3]. The paper by Hardy and Streifer [3] develops a new coupled mode formulation for the case of parallel waveguides by including the terms that are ignored in conventional theories, which is shown to give more accurate values for the propagation constants of the waveguide system and which is also suitable to be applied to more general systems of coupled waveguides.

The technique used here to formulate the problem of optical propagation in a coupled fibre-optic waveguide system is that of *path integration*, which was developed by Feynman to describe propagation of particles in a quantum mechanical system [9]. This technique has been used previously to study optical waveguides, in particular those structures of importance in integrated optics, by a number of authors and with considerable success (see [10–12] and references therein).

Propagation of light in the waveguides that are most suited to telecommunication systems is *paraxial* such that the wavefronts of the propagating electromagnetic wave are approximately planar [1]. Paraxial propagation of light in a dielectric fibre-optic waveguide can be described by a parabolic differential equation closely resembling the Schrödinger equation for motion of a quantum particle in a two-dimensional time-dependent potential well [13]. In order to construct the differential equation describing paraxial propagation in a waveguide structure, the refractive index distribution within the waveguides must be modelled in a physically realistic way. The conditions under which this description of optical propagation is applicable are particularly relevant to the waveguides that are used in fibre-optic devices because the refractive index of these waveguides varies slowly with position.

Feynman and Hibbs [9] showed that the *propagator* describing propagation in quantum mechanics can be written elegantly as a sum of terms, each of which describes propagation along a particular path in spacetime. The mathematical formulation of this sum over paths is referred to as a path integral. By exploiting the quantum mechanical analogue of the optical system, it is possible to write the propagator describing paraxial propagation as a sum of terms, each of which describes propagation along an optical path [11].

Describing optical propagation in waveguiding structures by means of a propagator written in terms of a path integral is fundamentally different from the other theoretical methods described above and there is an important distinction between the path integral method of studying optical propagation and the other theoretical techniques described above. The common starting point of the more conventional theoretical techniques is propagation in uncoupled waveguides which are separated by an infinite distance. The advantage of the path integral approach is that a propagator provides a global picture of propagation by describing the behaviour of all of the modes supported by the system simultaneously, which is fundamentally different from the approach of the other theoretical methods described earlier. As with quantum mechanics, the use of path integrals in optical propagation provides a new way of thinking about the problem and suggests new methods of solution.

In this paper, an existing sophisticated variational procedure due to Feynman and Kleinert [13,14], which is an extension of Feynman's original variational principle [9], is used to study the problem of propagation in the system consisting of two parallel coupled waveguides illustrated in figure 1. The parallel coupled waveguide system has been studied previously using a variational procedure developed by Constantinou and Jones [12]. This work follows and extends the previous calculations in [12] with a number of significant improvements. The variational procedure used in [12] was based on Feynman's original variational procedure [9]. However, this has been shown not to give good estimates for the energy levels of a quantum system under some circumstances [13].

The new variational procedure is shown to provide a better variational bound for the lowest-order propagation constant than the result of the calculation in [12]. It also predicts a definite distinction between the cases of strongly and weakly coupled waveguides and is able to make new predictions about the intermediate regime, which has previously proved inaccessible. An approximate propagator can be calculated using the variational procedure. From this propagator, the field profiles of the lowest-order and first excited

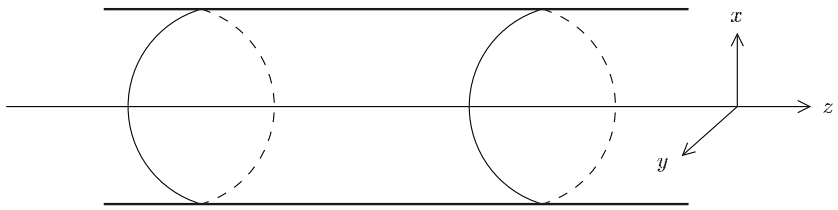


**Figure 2.** A typical fibre-optic guide, showing coordinate axes. The guide axis is chosen to coincide with the $z$-axis.

modes can be obtained in the strongly and weakly coupled limits. These field profiles are found to be more physically appealing than those calculated previously using a variational procedure [12].

The importance of analytical work such as that described in this paper is that it provides approximate but accurate analytical results which can be used to guide further experimental and numerical work on the design of integrated optical devices.

In section 2 we outline the basic formalism used in the calculation and in section 3 we describe in detail the variational procedure developed by Feynman and Kleinert [14] and used in this paper. Section 4 applies the variational technique to the coupled waveguide system, and in sections 5–7 we use the results obtained in the previous section to calculate quantities which are of interest in engineering. In section 5 we calculate approximate expressions for the propagation constant of the lowest-order mode of the system, in section 6 the field profile of the lowest-order mode of the system is calculated and in section 7 we estimate the propagation constants of the higher-order modes of the system and calculate the beat length of the parallel coupled waveguide system.

## 2. Formulation of problem

The technique used here to formulate the problem of optical propagation in a coupled fibre-optic waveguide system is that of *path integration*, which was developed by Feynman to describe propagation of particles in a quantum mechanical system [9]. In this section it will be shown that under certain conditions the optical propagation in a fibre-optic waveguide, such as that illustrated in figure 2, can be described by a path integral. For convenience the guide axis is chosen to coincide with the $z$-axis of the Cartesian coordinate system shown in figure 2.

By considering the propagation of a monochromatic electromagnetic wave in a weakly inhomogeneous dielectric medium in which the refractive index varies slowly with position on a scale of a typical wavelength of the propagating light, it can be shown that the wave equation for the Cartesian components of the electric or magnetic fields can be written in the form [7]

$$\nabla^2\phi - \varepsilon(x, y, z)\mu_0 \frac{\partial^2\phi}{\partial t^2} \approx 0 \tag{1}$$

where $\phi(x, y, z, t)$ is one of the Cartesian components of the electric or magnetic field and $\varepsilon(x, y, z)$ is the spatially varying permittivity of the medium.

Propagation of light in the waveguides that are most suited to telecommunication systems is *paraxial* such that the wavefronts of the propagating waves are approximately planar [1]. The general form of the refractive index in a fibre-optic waveguide means that the refractive index $n(x, y, z)$ can be expressed in the form [2]

$$n(x, y, z) = n_0\big[1 - n'(x, y, z)\big] \tag{2}$$

where $n'(x, y, z)$ is a smooth continuous function of position describing the spatial variation of the refractive index from its maximum value and will be referred to as the 'refractive index inhomogeneity function' [10]. The 'slowly varying envelope approximation' [5] is now used to obtain the form of equation (1) which is applicable to paraxial propagation. This is done by writing $\phi(x, y, z, t)$ as

$$\phi(x, y, z, t) = f(x, y, z)\, \exp\{\mathrm{i}(kz - \omega t)\} \tag{3}$$

where $f(x, y, z)$ is a slowly varying function of $z$ on a scale of $1/k$. The quantity $k$ in equation (3) is the maximum value of the wavenumber in the medium and is given by $k \equiv k_0 n_0$ in terms of the free-space wavenumber $k_0$. Using equation (1), it can be shown [10] that $f(x, y, z)$ satisfies

$$-\frac{1}{2k^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x, y, z) + n'(x, y, z)\, f(x, y, z) = \frac{\mathrm{i}}{k}\frac{\partial}{\partial z} f(x, y, z). \tag{4}$$

The parabolic differential equation given in equation (4) closely resembles the Schrödinger equation for motion of a quantum particle in a two-dimensional time-dependent potential well:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \psi(x, y, t) + V(x, y, t)\, \psi(x, y, t) = \mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x, y, t). \tag{5}$$

Feynman and Hibbs [9] showed that the *propagator* describing propagation in quantum mechanics can be written elegantly as a sum of terms, each of which describes propagation along a particular path in spacetime. By exploiting the quantum mechanical analogue of the optical system it is possible to write the propagator describing paraxial propagation as a sum of terms, each of which describes propagation along an optical path [11]. Using the analogies between paraxial optical propagation and quantum mechanical propagation, the 'optical propagator' can be written as a Feynman path integral [13]

$$\mathcal{K}(x_b, y_b, z_b; x_a, y_a, z_a) = \int_{(x_a, y_a, z_a)}^{(x_b, y_b, z_b)} \mathcal{D}x(z)\, \mathcal{D}y(z)\, \exp\left\{\mathrm{i}k\, \mathcal{S}_{a,b}\, [x(z), y(z)]\right\} \tag{6}$$

for $z_b > z_a$, and we have used the standard quantum mechanical notation for such objects. The quantity $\mathcal{S}_{a,b}\,[x(z), y(z)]$ in equation (6) is defined by

$$\mathcal{S}_{a,b}\,[x(z), y(z)] = \int_{z_a}^{z_b} \mathrm{d}z \left\{\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 - n'(x, y, z)\right\} \tag{7}$$

and will be referred to as the 'optical action'. The propagator defined by equation (6) describes the propagation of the quantity $f(x, y, z)$. It can be seen that the propagation of the Cartesian components of the physically relevant electric and magnetic fields of the light propagating in the waveguide system, which are related to $f(x, y, z)$ by equation (3), is described by the modified propagator

$$\tilde{\mathcal{K}}(x_b, y_b, z_b; x_a, y_a, z_a) = \int_{(x_a, y_a, z_a)}^{(x_b, y_b, z_b)} \mathcal{D}x(z)\, \mathcal{D}y(z)\, \exp\left\{\mathrm{i}k\, \tilde{\mathcal{S}}_{a,b}\, [x(z), y(z)]\right\} \tag{8}$$

where

$$\tilde{\mathcal{S}}_{a,b}\,[x(z), y(z)] = (z_b - z_a) + \mathcal{S}_{a,b}\,[x(z), y(z)] \tag{9}$$

and in the paraxial approximation for the refractive index given in equation (2), is proportional to the *optical path length* [7, 15].

In order to construct the propagator describing paraxial propagation in a waveguide structure, the refractive index distribution within the waveguides must be modelled in a physically realistic way.

This paper describes the use of a sophisticated variational technique [13, 14] to calculate approximate expressions for the propagation constants and modal field profiles of the modes supported by the system consisting of two parallel coupled waveguides. The results are compared with those of a previous variational calculation by Constantinou and Jones [12] which made use of a less sophisticated variational procedure based on Feynman's original variational technique [9].

The refractive index distribution used both here and by Constantinou and Jones [12] to model the system consisting of two parallel coupled waveguides is given by

$$n(x, y, z) = n_0\big[1 - a^4\big(x^2 - b^2\big)^2 - \tfrac{1}{2}\omega_y^2 y^2\big] \tag{10}$$

where $a$, $b$ and $\omega_y$ are constants. Figure 3 compares the variation in the $x$-direction of this choice of model refractive index with the expected distribution to be found in the real fibre system. The distribution was chosen to resemble the physical reality of the system as closely as possible as shown in figure 3. The model refractive index distribution given by equation (10) is suitable for modelling the situation when the waveguides are close together and far apart because of the local minimum in the distribution at $x = 0$. The depth of this minimum, measured from $n_0$, is $n_0a^4b^4$. When the separation of the two guides is small compared with their width, there is significant overlap of the fields centred on each guide and the guides are strongly coupled. This is taken into account by the model distribution because, in this situation, $b$ is small and the local minimum is very shallow. When the separation of the guides is large compared with their width, there is negligible overlap of the fields centred on each guide and the guides are weakly coupled. In this situation, $b$ is large and the local minimum is very deep. The model refractive index distribution given by equation (10) is also suitable for modelling the refractive index distribution found in an integrated optical device consisting of two parallel waveguides. Such waveguides are formed on a single dielectric substrate using the technique of metal in-diffusion [1]. This process produces a refractive index which varies smoothly in the $x$-direction, with the refractive index between the two waveguides closely resembling the shape of the local minimum at $x = 0$ of the model distribution given by equation (10).

Using the forms of the propagator given in equation (6) and the refractive index distribution of equation (10), the two-dimensional propagator in equation (6) separates into the product of two one-dimensional propagators. The resulting one-dimensional propagator, which depends on the $x$-coordinate, is given by

$$\mathcal{K}_x(x_b, z_b; x_a, z_a) = \int_{(x_a,z_a)}^{(x_b,z_b)} \mathcal{D}x(z)\, \exp\left\{ \mathrm{i}k \int_{z_a}^{z_b} \mathrm{d}z \left[ \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 - n'_x(x, z) \right] \right\} \tag{11}$$

where $n'_x(x, z)$ is the part of the refractive index inhomogeneity function which depends on the $x$-coordinate, and will be referred to as the 'partial refractive index inhomogeneity function'. For the case of the model distribution described by equation (10), $n'_x(x, z)$ is independent of the $z$-coordinate and is given by

$$n'_x(x, z) \equiv V(x) = a^4\big(x^2 - b^2\big)^2. \tag{12}$$

As in the quantum mechanical case, the propagator given by equation (8) for the system with the refractive index distribution given by equation (10) can be expanded in terms of the eigenfunctions and eigenvalues of the system as [9]

$$\tilde{\mathcal{K}}(x_b, y_b, z_b; x_a, y_a, z_a) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varphi_{n,m}(x_b, y_b)\, \varphi^*_{n,m}(x_a, y_a)\, \exp\big\{\mathrm{i}\beta_{n,m}\,(z_b - z_a)\big\} \tag{13}$$

for $z_b > z_a$, where $\varphi_{n,m}(x, y)$ is the mode field profile and $\beta_{n,m}$ is the propagation constant of the mode of the system labelled by $(n, m)$ [7]. For the two-dimensional propagator given

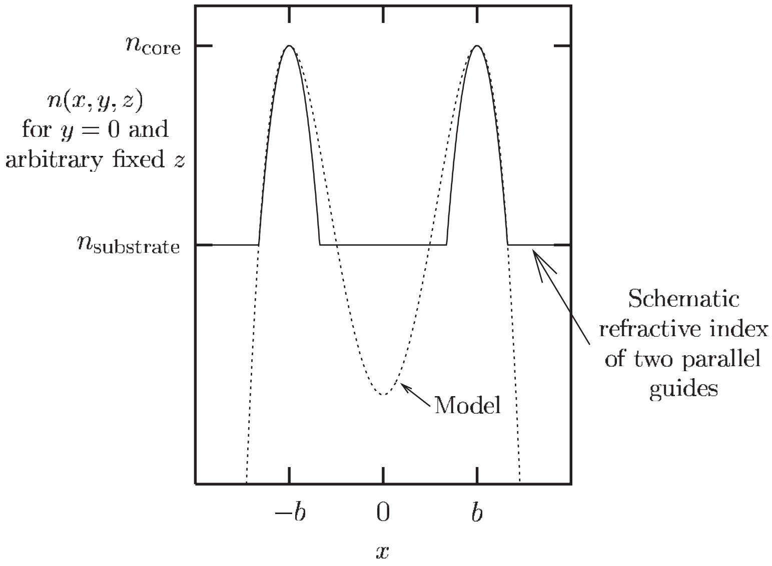


**Figure 3.** The refractive index profile $n(x, y, z)$ for $y = 0$ and arbitrary fixed $z$ of two parallel waveguides. The dotted curve shows the model distribution given in equation (10), with $n_0 \equiv n_{\text{core}}$, and the solid curve shows the schematic refractive index distribution across two parallel guides.

by equation (6), the expression for $\tilde{\mathcal{K}}$ separates into the product of two one-dimensional propagators having the field profiles $\varphi_n^x(x)$ and $\varphi_m^y(y)$ and propagation constants $\beta_n^x$ and $\beta_m^y$ so that

$$\tilde{\mathcal{K}}(x_b, y_b, z_b; x_a, y_a, z_a) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varphi_n^x(x_b)\,\varphi_m^y(y_b)\,\varphi_n^{x*}(x_a)\,\varphi_m^{y*}(y_a) \times \exp\left\{\mathrm{i}\left[k - \beta_n^x - \beta_m^y\right](z_b - z_a)\right\} \tag{14}$$

for $z_b > z_a$. The propagator given by equation (11) can be similarly expanded in terms of the partial mode field profiles $\varphi_n^x(x)$ and partial propagation constants $\beta_n^x$ of the one-dimensional system as [9]

$$\mathcal{K}_x(x_b, z_b; x_a, z_a) = \sum_{n=0}^{\infty} \varphi_n^x(x_b)\,\varphi_n^{x*}(x_a)\,\exp\left\{-\mathrm{i}\beta_n^x\,(z_b - z_a)\right\} \tag{15}$$

for $z_b > z_a$. In the analogous quantum mechanical problem, $\beta_n^x$ would be the energy levels and $\varphi_n^x(x)$ would be the corresponding wavefunctions.

In the previous coupled waveguide variational calculation by Constantinou and Jones [12], the $y$-dependence of the system is omitted so that the mode field profile and the propagation constant of the mode labelled by $(n, m)$ become

$$\varphi_n(x) \equiv \varphi_{n,m}(x, y) = \varphi_n^x(x) \qquad \text{with} \quad \beta_n \equiv \beta_{n,m} = k - \beta_n^x \tag{16}$$

which latter are the eigenfunctions and ‘eigenvalues’ used in equation (15).

## 3. Variational procedure

In the paper by Feynman and Kleinert [14], a variational procedure is developed and used to accurately calculate the free energy of a quantum system as a function of temperature. From this free energy, it is possible to calculate an approximate value for the ground state energy of the quantum system. Using the analogies between the quantum mechanical problem and the optical problem outlined in section 2, it can be seen that the partial propagation constant $\beta_n^x$ defined by equation (15) corresponds to $E_n/\hbar$, where $E_n$ is the energy level of the analogous quantum mechanical system. This means that the variational procedure developed by Feynman and Kleinert [14] can also be used to obtain an approximation for the partial propagation constant $\beta_n^x$ of the optical system. In this section the variational procedure of [14] will be formulated in terms of the parameters of the optical problem.

The starting point for the calculation is the one-dimensional propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$ given by equation (11). For a system with a refractive index distribution which does not depend on $z$ explicitly (which is the case for the system with the refractive index distribution given by equation (10)), the propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$ depends only on the difference $(z_b - z_a)$, so that $\mathcal{K}_x(x_b, z_b; x_a, z_a) = \mathcal{K}_x(x_b, z_b - z_a; x_a, 0)$. By analogy with the quantum mechanical problem [9], the quantity $(z_b - z_a)$ in the propagator is replaced by $-\mathrm{i}\mu/k$, where $\mu$ is real and will be referred to as the 'imaginary propagation distance' of the problem. Using this imaginary propagation distance, a quantity $Z$ is defined in terms of the one-dimensional propagator $\mathcal{K}_x$ by the equation

$$Z \equiv \int \mathrm{d}x_a\, \mathcal{K}_x(x_a, -\mathrm{i}\mu; x_a, 0) \tag{17}$$

which uses units with $k = 1$. The quantity $Z$ is the analogue, for the optical system, of the partition function of the analogous quantum mechanical problem [13]. It will be referred to as the *optical partition function*. From the path integral representation of the propagator $\mathcal{K}_x$ given by equation (11) and for the case where the partial refractive index inhomogeneity function $n'_x(x, z)$ is independent of $z$, and can be written as $n'_x(x, z) \equiv V(x)$, the optical partition function $Z$ can be expressed as [13]

$$Z \equiv \oint \mathcal{D}x(\zeta) \exp\left\{ -\int_0^\mu \mathrm{d}\zeta \left[\tfrac{1}{2}\dot{x}^2(\zeta) + V\big(x(\zeta)\big)\right]\right\} \tag{18}$$

where $\dot{x}(\zeta) \equiv \mathrm{d}x(\zeta)/\mathrm{d}\zeta$ and the path integral represented by $\oint \mathcal{D}x(\zeta)\ldots$ is an integral over all paths which start and finish at the same value of the $x$-coordinate, and includes an integration over this end-point, as shown in equation (17). The integral $\int_0^\mu \mathrm{d}\zeta\left[\frac{1}{2}\dot{x}^2(\zeta) + V\big(x(\zeta)\big)\right]$ in equation (18) will be called the *optical Euclidean action* since it is the Euclidean action in the analogous quantum mechanical system [13]. The path integral of equation (18) is most easily handled if the paths $x(\zeta)$ in the path integral are written as a Fourier series [13]. We denote the zero-frequency Fourier component, which measures the average of $x(\zeta)$, by $x_0$.

In the limit $\mu \to +\infty$ the integrations with respect to the nonzero frequency Fourier components in the optical partition function (18) can be evaluated with the result that [13]

$$Z \xrightarrow[\mu\to+\infty]{} \int_{-\infty}^{+\infty} \frac{\mathrm{d}x_0}{\sqrt{2\pi\mu}}\, \mathrm{e}^{-\mu V(x_0)} \tag{19}$$

which is the ray optics limit of the optical partition function.

If the integrations with respect to the nonzero frequency Fourier components in the optical partition function (18) could be evaluated for any value of $\mu$, the resulting expression for $Z$ would be a single integral over the Fourier (average) component $x_0$ of the form [14]

$$Z = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x_0}{\sqrt{2\pi\mu}}\, \mathrm{e}^{-\mu W(x_0)}. \tag{20}$$

In the context of statistical mechanics the function $W(x_0)$ is conventionally referred to as an effective classical potential [14]. Here, in the context of ray optics, $W(x_0)$ is an effective partial refractive index inhomogeneity function since equation (20) has the same form as the ray optics limit of the optical partition function given by equation (19). $W(x_0)$ can be thought of as the partial refractive index inhomogeneity function which when used in the ray optics description of the problem (with a ray path for which $\int_0^\mu \mathrm{d}\zeta\, x(\zeta) = \mu x_0$) provides the same values for the physical quantities as the wave optics description of the problem. We shall refer to $W(x_0)$ as the effective refractive index.

An upper bound for $W(x_0)$ can be calculated by introducing a trial optical partition function in which that part of the partial refractive index inhomogeneity function $V(x)$ depending on the nonzero frequency Fourier components is replaced by a quadratic function whose maximum is at $x_0$ and whose curvature depends on $x_0$. Such a trial optical partition function is given by

$$Z^{(1)} = \oint \mathcal{D}x(\zeta) \exp\left\{ - \int_0^\mu \mathrm{d}\zeta \left[\tfrac{1}{2}\dot{x}^2(\zeta) + \tfrac{1}{2}\Omega^2(x_0)\,[x(\zeta) - x_0]^2\right] - \mu L^{(1)}(x_0)\right\} \tag{21}$$

where $\Omega(x_0)$ is an arbitrary local curvature of the partial refractive index inhomogeneity function and $L^{(1)}(x_0)$ is a part of the partial refractive index inhomogeneity function depending only on the average coordinate $x_0$.

By analogy with equation (20), $Z^{(1)}$ can be written as [14]

$$Z^{(1)} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x_0}{\sqrt{2\pi\mu}} \exp\left\{ - \mu \tilde{W}^{(1)}\big(x_0, a^2(x_0), \Omega(x_0)\big)\right\} \tag{22}$$

where the function $\tilde{W}^{(1)}\big(x_0, a^2, \Omega\big)$ is determined by optimizing $Z^{(1)}$ with respect to $\Omega(x_0)$ and $L^{(1)}(x_0)$. Some algebra shows that

$$\tilde{W}^{(1)}\big(x_0, a^2, \Omega\big) = \frac{1}{\mu} \ln\left[\frac{\sinh(\mu\,\Omega/2)}{\mu\,\Omega/2}\right] - \frac{1}{2}\Omega^2 a^2 + V_{a^2}(x_0) \tag{23}$$

where

$$V_{a^2(x_0)}(x) \equiv \int_{-\infty}^{+\infty} \frac{\mathrm{d}x'}{\big(2\pi a^2(x_0)\big)^{1/2}} \exp\left\{-\frac{1}{2a^2(x_0)}\big(x - x'\big)^2\right\} V\big(x'\big)\,. \tag{24}$$

We see that $V_{a^2}(x)$ is a kind of smeared version of the partial refractive index inhomogeneity function $V(x)$, as described by Feynman and Kleinert [14]; the smearing width $a(x_0)$ is given by

$$a^2(x_0) = \frac{1}{\mu\,\Omega^2(x_0)}\left[\frac{\mu\,\Omega(x_0)}{2} \coth\left(\frac{\mu\,\Omega(x_0)}{2}\right) - 1\right]. \tag{25}$$

Using equation (23) and the Jensen–Peierls variational inequality [13] we find that

$$W(x_0) < \tilde{W}^{(1)}\big(x_0, a^2(x_0), \Omega(x_0)\big) \tag{26}$$

so that $W(x_0)$ is bounded from above locally by $\tilde{W}^{(1)}\big(x_0, a^2(x_0), \Omega(x_0)\big)$. This bound can be optimized by minimizing $\tilde{W}^{(1)}\big(x_0, a^2(x_0), \Omega(x_0)\big)$ with respect to the parameters $a^2(x_0)$ and $\Omega(x_0)$, yielding a minimum value $W^{(1)}(x_0)$ given by

$$W^{(1)}(x_0) = \min_{a^2(x_0),\Omega(x_0)} \left\{\tilde{W}^{(1)}\big(x_0, a^2(x_0), \Omega(x_0)\big)\right\}. \tag{27}$$

The minimization indicated in equation (27) is carried out subject to the condition that $a^2(x_0)$ and $\Omega(x_0)$ are related by equation (25) and leads to a second equation relating $a^2(x_0)$ and $\Omega(x_0)$ of the form

$$\Omega^2(x_0) = 2\frac{\partial}{\partial a^2} V_{a^2}(x_0) = \frac{\partial^2}{\partial x_0^2} V_{a^2}(x_0)\,. \tag{28}$$

From this we see that

$$W^{(1)}(x_0) = \tilde{W}^{(1)}\big(x_0, a^2_{\min}(x_0), \Omega_{\min}(x_0)\big) \tag{29}$$

where $a^2_{\min}(x_0)$ and $\Omega_{\min}(x_0)$ satisfy equations (25) and (28) for each value of $x_0$.

An approximation to the partial propagation constant of the lowest-order mode $\beta^x_0$ of the optical system can be obtained from the approximate effective refractive index $W^{(1)}(x_0)$ in a way which is analogous to that described by Feynman and Kleinert [14]. Using the expansion of the propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$ in terms of partial mode field profiles $\varphi^x_n(x)$ and partial propagation constants $\beta^x_n$ given by equation (15), and the expression for $Z$ given by equation (17), it can be seen that the optical partition function $Z$ can be written as

$$Z = \sum_{i=0}^{\infty} \exp\big\{-\mu\beta^x_i\big\} \tag{30}$$

since the partial mode field profiles can be chosen to be an orthonormal set.

If the partial propagation constants $\big\{\beta^x_n\big\}$ ($n = 0, 1, \ldots$) are ordered so that $\beta^x_0 < \beta^x_1 < \cdots$, then when we take the limit $\mu \to +\infty$ of the optical partition function in equation (30) we find that the partial propagation constant, $\beta^x_0$, of the lowest-order mode is determined by

$$\lim_{\mu\to+\infty} Z = \exp\big\{-\mu\beta^x_0\big\} \tag{31}$$

so that an approximation to the lowest-order mode partial propagation constant $\beta_0^{x(1)}$ is obtained by minimizing the function $W^{(1)}(x)$ with respect to $x$ in the limit $\mu \to +\infty$. We find that [14]

$$\beta_0^{x\,(1)} = \min_x\Big\{\lim_{\mu\to+\infty} W^{(1)}(x)\Big\} = \min_x\left\{V_{a^2_{\min}(x)}(x) + \frac{1}{8a^2_{\min}(x)}\right\}. \tag{32}$$

In the limit $\mu \to +\infty$ the approximate effective refractive index becomes

$$\lim_{\mu\to+\infty} W^{(1)}(x) = V_{a^2_{\min}(x)}(x) + \frac{1}{8a^2_{\min}(x)} \tag{33}$$

with $a^2_{\min}(x)$ determined by combining equations (25) and (28) in the limit $\mu \to +\infty$. This implies that the approximation to the lowest-order mode partial propagation constant $\beta_0^{x\,(1)}$ given by equation (32) is

$$\beta_0^{x\,(1)} = \min_x\left\{V_{a^2_{\min}(x)}(x) + \frac{1}{8a^2_{\min}(x)}\right\}. \tag{34}$$

In sections 4 and 5 we apply the procedure described above to the system consisting of two parallel coupled waveguides and use the result given by equation (34) to calculate an approximate expression for the propagation constant of the lowest-order mode of the system.

It should also be noted that if we take the limit $\mu \to 0^+$ the approximate effective refractive index, $W^{(1)}(x)$, coincides with $V(x)$ [13], so that

$$\lim_{\mu\to 0^+} W^{(1)}(x) = V(x). \tag{35}$$

However, we do not consider the significance of this result in any detail in this paper.

The variational procedure that has been described in this section is different from that which has been used previously to study optical propagation in the parallel coupled waveguide system by Constantinou and Jones [12]. Approximate expressions for the propagation constant and field profile of the lowest-order mode of the parallel coupled waveguide system were those found by calculating an approximate closed-form expression for the propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$ using a trial propagator $\mathcal{K}^{\text{trial}}_x(x_b, z_b; x_a, z_a)$ which is of the form

$$\mathcal{K}^{\text{trial}}_x(x_b, z_b; x_a, z_a) = \int_{(x_a,z_a)}^{(x_b,z_b)} \mathcal{D}x \exp\left\{\mathrm{i}k\int_{z_a}^{z_b}\left[\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}c^2x^2\right]\mathrm{d}z\right\} \tag{36}$$

and which involves one variational parameter $c$, the curvature of the trial partial refractive index inhomogeneity function [12]. By writing the approximate expression for the propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$ in terms of the imaginary propagation distance $\mu$ via $z_b - z_a = -i\mu/k$ and taking the limit $\mu \to +\infty$, expressions for the propagation constant and field profile of the lowest-order mode were obtained in terms of the variational parameter $c$. These expressions were then optimized by performing variations with respect to the parameter $c$. This variational procedure is expected to provide approximations which are less accurate than those obtained by the variational procedure described above, based on the work of Feynman and Kleinert, since only one variational parameter is used. In addition the form of the trial partial refractive index inhomogeneity function that is used by Constantinou and Jones is only expected to provide good approximations for strongly coupled waveguides [12] because, as discussed in section 1, for the case of strong coupling the model refractive index given by equation (10) has a very shallow minimum at $x = 0$ so that the partial refractive index inhomogeneity function $V(x)$ has a single minimum at $x = 0$ and is approximated well by a quadratic function.

## 4. Application to the parallel coupled waveguide system

The procedure described in section 3 will now be applied to the parallel coupled waveguide system introduced previously.

The refractive index distribution of the parallel coupled waveguide system described in section 2 is modelled by the distribution given by equation (10), with the partial refractive index inhomogeneity function given by equation (12) as

$$V(x) = a^4\left(x^2 - b^2\right)^2. \tag{37}$$

The smeared partial refractive index inhomogeneity function corresponding to the partial refractive index inhomogeneity function $V(x)$ is given by

$$V_{\alpha^2}(x) = \int_{-\infty}^{+\infty} \frac{dx'}{\left(2\pi\alpha^2\right)^{1/2}} \exp\left\{-\frac{1}{2\alpha^2}\left(x - x'\right)^2\right\} V(x'). \tag{38}$$

Substituting equation (37) into (38) gives

$$V_{\alpha^2}(x) = a^4x^4 + 2a^4\left(3\alpha^2 - b^2\right)x^2 + a^4\left(b^4 + 3\alpha^4 - 2b^2\alpha^2\right). \tag{39}$$

The approximate effective refractive index, $W^{(1)}(x)$, for this system is given by

$$W^{(1)}(x_0) = \tilde{W}^{(1)}\left(x_0, \alpha^2_{\min}(x_0), \Omega_{\min}(x_0)\right) \tag{40}$$

where

$$\tilde{W}^{(1)}\left(x_0, \alpha^2, \Omega\right) = \frac{1}{\mu}\ln\left[\frac{\sinh\left(\mu\,\Omega/2\right)}{\mu\,\Omega/2}\right] - \frac{1}{2}\Omega^2\alpha^2 + V_{\alpha^2}(x_0)\,. \tag{41}$$

The quantities $\alpha^2_{\min}(x_0)$ and $\Omega_{\min}(x_0)$ in equation (40) are determined by carrying out the minimization described in section 3. Using equations (28) and (25) for this case shows that the values of $\alpha^2_{\min}(x_0)$ and $\Omega_{\min}(x_0)$ satisfy the equations

$$\Omega^2(x_0) = 4a^2\left[3\left(\alpha^2(x_0) + x_0^2\right) - b^2\right] \tag{42}$$

and

$$\alpha^2(x_0) = \frac{1}{\mu\,\Omega^2(x_0)}\left[\frac{\mu\,\Omega(x_0)}{2}\coth\left(\frac{\mu\,\Omega(x_0)}{2}\right) - 1\right] \tag{43}$$

for each value of $x_0$. There are a number of problems encountered when equations (42) and (43) are solved for a general value of $\mu$ [13]. However the approximate effective refractive index

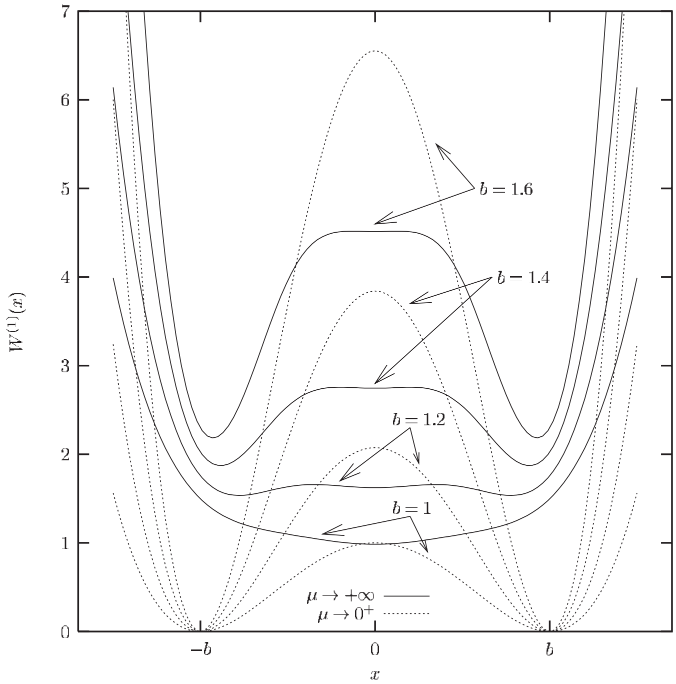


**Figure 4.** The approximate effective refractive index $W^{(1)}(x)$ in the limits $\mu \to +\infty$ (solid curve) and $\mu \to 0^+$ (dotted curve) for the partial refractive index inhomogeneity function $V(x) = a^4(x^2 - b^2)^2$ at various values of $b$ with $a = 1$. At $b = b_c$ the stationary values at $x \neq 0$ of the $\mu \to +\infty$ limit of $W^{(1)}(x)$ appear. For $b < \tilde{b}_c$ the minimum of the $\mu \to +\infty$ limit of $W^{(1)}(x)$ lies at $x = 0$ and for $b > \tilde{b}_c$ there are two minima at $x = \pm x_{\text{stat}} \neq 0$. For $a = 1$, $b_c \approx 1.113$ and $\tilde{b}_c \approx 1.159$.

can be found easily in the two limits $\mu \to 0^+$ and $+\infty$. The latter was shown in section 3 to give the lowest-order propagation constant of the system. In the limit $\mu \to 0^+$ the approximate effective refractive index, $W^{(1)}(x)$, coincides with the partial refractive index inhomogeneity function $V(x)$ as shown in equation (35). In the limit $\mu \to +\infty$, equation (43) becomes $\alpha^2(x_0) = 1/(2\Omega(x_0))$, so that using equation (42) it can be seen that $\alpha^2_{\text{min}}(x_0)$ satisfies the equation

$$3\left(\alpha^2_{\text{min}}(x_0)\right)^3 + \left(3x^2 - b^2\right)\left(\alpha^2_{\text{min}}(x_0)\right)^2 - \frac{1}{16a^4} = 0 \tag{44}$$

and that the expression for the approximate effective refractive index, in the limit $\mu \to +\infty$, is given by

$$\lim_{\mu \to +\infty} W^{(1)}(x) = V_{\alpha^2_{\text{min}}(x)}(x) + \frac{1}{8\alpha^2_{\text{min}}(x)}. \tag{45}$$

Although equation (44) can be solved analytically, we choose to solve it numerically by iteration for each value of $x$. The approximate effective refractive index in the limit $\mu \to +\infty$ obtained using the numerical solution of equation (44) is plotted in figure 4 for various values of the parameter $b$. The approximate effective refractive index in the limit $\mu \to 0^+$ for the same values of the parameter $b$ is also plotted in figure 4.

## 5. Estimation of the propagation constant of the lowest-order mode

It has been shown in section 3 that the minimum of the approximate effective refractive index, $W^{(1)}(x)$, in the limit $\mu \to +\infty$ gives an approximation to the propagation constant of the lowest-order mode of a waveguiding system. It has also been shown that this approximation is a lower bound for the actual propagation constant of the lowest-order mode of the system.

The approximation to the partial propagation constant of the lowest-order mode $\beta_0^x$ obtained from the variational calculation is obtained by minimizing with respect to $x$ the limit of the approximate effective refractive index as $\mu \to +\infty$ as shown in equation (34). Using this equation and the expression in equation (45), the value of $\beta_0^{x\,(1)}$ can be written as

$$\beta_0^{x\,(1)} = \min_x \left\{ V_{\alpha^2}(x) + \frac{1}{8\alpha^2} \right\}. \tag{46}$$

The quantity in equation (46) which needs to be minimized is given by

$$\tilde{V}(x) = V_{\alpha^2}(x) + \frac{1}{8\alpha^2} = a^4x^4 + 2a^4\big(3\alpha^2 - b^2\big)x^2 + a^4\big(b^4 + 3\alpha^4 - 2b^2\alpha^2\big) + \frac{1}{8\alpha^2} \tag{47}$$

where we have used the smeared version of the partial refractive index inhomogeneity function $V(x) = a^4\left(x^2 - b^2\right)^2$ given by equation (39).

Carrying out the minimization of $\tilde{V}(x)$ shows that the minimum of $\tilde{V}(x)$ occurs at either

$$x = 0 \qquad \text{or} \qquad x = \pm x_{\text{stat}} \neq 0 \tag{48}$$

where

$$x_{\text{stat}} = \sqrt{b^2 - 3\alpha^2} = \sqrt{b^2 - \frac{3}{2\Omega}}. \tag{49}$$

It is found that the value of $\tilde{V}(x)$ at $x = 0$ is

$$\tilde{V}(0) = a^4\left(b^4 + \frac{3}{4\Omega_1^2} - \frac{b^2}{\Omega_1}\right) + \frac{\Omega_1}{4} \tag{50}$$

where $\Omega_1$ satisfies the equation

$$\Omega_1^3 + 4a^4b^2\Omega_1 - 6a^4 = 0. \tag{51}$$

It can also be readily seen that

$$\tilde{V}(\pm x_{\text{stat}}) = \frac{a^4}{\Omega_2}\left(2b^2 - \frac{3}{2\Omega_2}\right) + \frac{\Omega_2}{4} \tag{52}$$

where $\Omega_2$ satisfies the equation

$$\Omega_2^3 - 8a^4b^2\Omega_2 + 12a^4 = 0. \tag{53}$$

Equation (51) has one real root given by

$$\Omega_1^* = \big(3a^4\big)^{1/3}\big\{\big[\big(1 + \tfrac{64}{243}a^4b^6\big)^{1/2} + 1\big]^{1/3} - \big[\big(1 + \tfrac{64}{243}a^4b^6\big)^{1/2} - 1\big]^{1/3}\big\} \tag{54}$$

and two complex roots for all physical values of $a$ and $b$. The two complex roots will be ignored since they are unphysical. The value of $\tilde{V}(x)$ at the local minimum at $x = 0$ is given by

$$\tilde{V}_{\min}(0) = a^4\left(b^4 + \frac{3}{4\Omega_1^{*2}} - \frac{b^2}{\Omega_1^*}\right) + \frac{\Omega_1^*}{4} \tag{55}$$

with $\Omega_1^*$ given by equation (54).

Equation (53) only has physically acceptable roots (i.e. roots which are real and positive) for $b > b_c$ where

$$b_c = \left(\frac{243}{128a^4}\right)^{1/6}. \tag{56}$$

For $a = 1$, $b_c$ has the numerical value 1.113. This means that the minimum of $\tilde{V}(x)$ occurs at $x = 0$ for $b < b_c$, as can be seen from the plots in figure 4, because the only physically acceptable solution for the minimum occurs at $x = 0$.

For $b > b_c$, equation (53) has two real roots, which give physically acceptable values of the $\mu \to +\infty$ limit of the approximate effective refractive index, $W^{(1)}(x)$. These values correspond to the stationary values of $\tilde{V}(x)$ which can be seen in figure 4 for $b > b_c$. The real root of equation (53) which corresponds to the local minima of $\tilde{V}(x)$ is

$$\Omega_2^* = 4\sqrt{\frac{3}{2}}\, a^2 b \cos\left(\frac{1}{3}\arccos\left[-\left(\frac{b_c}{b}\right)^3\right]\right) \tag{57}$$

so that the value of $\tilde{V}(x)$ at the local minima with $x \neq 0$ ($x = \pm x_{\text{stat}}$) is given by

$$\tilde{V}_{\min}(\pm x_{\text{stat}}) = \frac{a^4}{\Omega_2^*}\left(2b^2 - \frac{3}{2\Omega_2^*}\right) + \frac{\Omega_2^*}{4} \tag{58}$$

with $\Omega_2^*$ given by equation (57).

We can use equations (50) and (52) to solve $\tilde{V}(0) = \tilde{V}(\pm x_{\text{stat}})$ numerically and we find that the minimum of $\tilde{V}(x)$ lies at $x = 0$ for $b < \tilde{b}_c$ and lies at $x = \pm x_{\text{stat}}$ for $b > \tilde{b}_c$, where $\tilde{b}_c$ has the numerical value 1.159 when $a = 1$.

In summary, as $b$ is increased from zero, the $\mu \to +\infty$ limit of $W_1(x)$ (which is denoted by $\tilde{V}(x)$ and given by equation (47)) has a single minimum at $x = 0$ for $b < b_c$. At $b = b_c$ subsidiary minima of $\tilde{V}(x)$ appear at $x = \pm x_{\text{stat}}$, so that for $b > b_c$ $\tilde{V}(x)$ has three local minima. At $b = \tilde{b}_c$ the minima at $x = \pm x_{\text{stat}}$ have the same height as the minimum at $x = 0$ (i.e. $\tilde{V}(0) = \tilde{V}(\pm x_{\text{stat}})$). The minima at $x = \pm x_{\text{stat}}$ are lower than the minimum at $x = 0$ for $b > \tilde{b}_c$. This behaviour is illustrated in figure 4. From this, the approximation to the partial propagation constant of the lowest-order mode $\beta_0^x$ obtained by the variational calculation is

$$\beta_0^x \approx \begin{cases} \tilde{V}_{\min}(0) & \text{for} \quad b < \tilde{b}_c \\ \tilde{V}_{\min}(\pm x_{\text{stat}}) & \text{for} \quad b > \tilde{b}_c \end{cases} \tag{59}$$

with $\tilde{V}_{\min}(0)$ and $\tilde{V}_{\min}(\pm x_{\text{stat}})$ given by equations (55) and (58), respectively.

Using equation (16), it can be seen that the partial propagation constant of the lowest-order mode given by equation (59) corresponds to a value of $\beta_0$, the propagation constant of the lowest-order mode of the system, given by

$$\beta_0 \approx \begin{cases} k - \dfrac{\Omega_<^*}{4} + \dfrac{a^4 b^2}{\Omega_<^*} - k a^4 b^4 - \dfrac{3a^4}{4k\Omega_<^{*2}} & \text{for} \quad b < \tilde{b}_c \\[2ex] k - \dfrac{\Omega_>^*}{4} - \dfrac{2a^4 b^2}{\Omega_>^*} + \dfrac{3a^4}{2k\Omega_>^{*2}} & \text{for} \quad b > \tilde{b}_c \end{cases} \tag{60}$$

where $\Omega_<^*$ and $\Omega_>^*$ are given by

$$\Omega_<^* = \left(\frac{3a^4}{k}\right)^{1/3}\left\{\left[\left(1 + \frac{64}{243}k^2 a^4 b^6\right)^{1/2} + 1\right]^{1/3} - \left[\left(1 + \frac{64}{243}k^2 a^4 b^6\right)^{1/2} - 1\right]^{1/3}\right\} \tag{61}$$

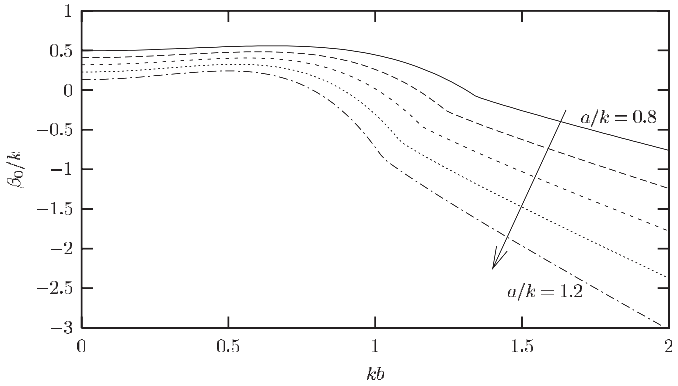


**Figure 5.** The dimensionless approximate lowest-order mode propagation constant $\beta_0/k$ given by equation (60) as a function of the dimensionless guide separation $kb$ for $a/k = 0.8, 0.9, 1, 1.1$ and $1.2$, respectively.

and

$$\Omega^*_> = 4\sqrt{\frac{3}{2}}\,a^2b\,\cos\left(\frac{1}{3}\arccos\left[-\frac{9}{8}\sqrt{\frac{3}{2}}\frac{1}{ka^2b^3}\right]\right). \tag{62}$$

The critical value of $b$, $\tilde{b}_c$, can be calculated numerically in the same way as described above. The value of $\tilde{b}_c$ for the (arbitrary) case of $a = k$ is $\tilde{b}_c \approx 1.159k^{-1}$. The approximate propagation constant of the lowest-order mode is plotted in figure 5 as a function of the dimensionless guide separation $kb$ for a number of values for the dimensionless potential parameter $a/k$.

The expression for the approximate propagation constant of the lowest-order mode obtained by the previous variational calculation by Constantinou and Jones [12] (which was mentioned at the end of section 3) and valid in the strong-coupling limit is

$$\beta_0^{\mathrm{C-J}} \approx k - \frac{c}{2} + \left[\frac{a^4b^2}{c^2} + \frac{1}{4}\right]c - ka^4b^4 - \frac{3a^4}{4kc^2} \tag{63}$$

with the value of $c$ given by

$$c = \left(\frac{3a^4}{2k}\right)^{1/3}\left\{\left[\left(1+\left(\frac{2}{3}\right)^5k^2a^4b^6\right)^{1/2}+1\right]^{1/3} - \left[\left(1+\left(\frac{2}{3}\right)^5k^2a^4b^6\right)^{1/2}-1\right]^{1/3}\right\}. \tag{64}$$

The expression given by equation (63) is compared with the result of the variational calculation described in this paper (equation (60)) in figure 6.

The use of a variational technique implies that the true result for the lowest-order mode propagation constant always lies above the approximate value. Figure 6 shows that since our present result lies above the previous result of Constantinou and Jones it must lie closer to the exact value of $\beta_0$. Strong coupling between the guides occurs when there is significant overlap of the fields centred on each guide. This physically occurs when the guides are close together and from equation (10) will occur when $kb < 1$, when the refractive index has a shallow minimum at $x = 0$. Conversely, the guides are weakly coupled if the overlap of fields centred in the individual guides is very small, and this corresponds in our model to $kb > 1$ when the refractive index has a very deep minimum at $x = 0$. The intermediate regime, previously inaccessible, occurs when $kb \sim 1$. The previous variational result due to Constantinou and

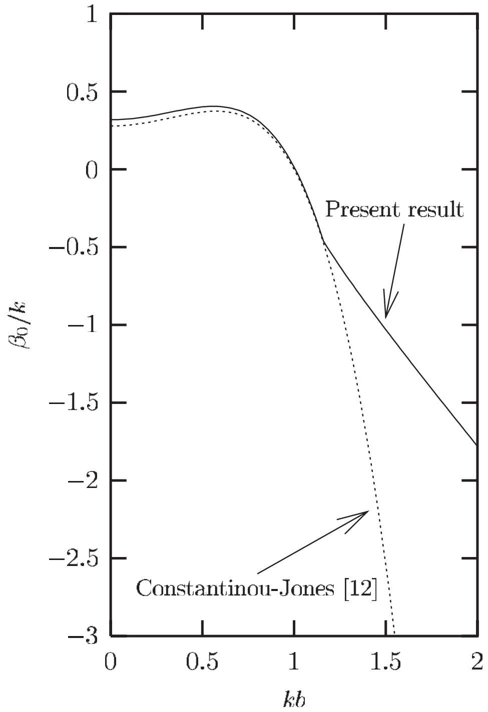


**Figure 6.** The dimensionless approximate lowest-order mode propagation constant $\beta_0/k$ given by equation (60) as a function of the dimensionless guide separation $kb$ with $a = k$, compared with the result obtained by Constantinou and Jones [12] and given by equation (63).

Jones was only valid in the strong-coupling limit, but the new result is expected to be valid for both strong and weak coupling of the waveguides [14]. In figure 6 we see that the propagation constant $\beta_0$ as a function of $kb$ changes its shape when $kb \approx 1.159$. Although the apparent discontinuity in the derivative of $\beta_0$ is a consequence of the mathematical model we have used, nevertheless it is gratifying to see this anomaly occurring at the value of $b$ at which we expect that we are leaving the strong-coupling regime and entering the previously unexplored intermediate regime.

The authors have also used a numerical technique (to be published) to calculate values for the propagation constants and field profiles of the waveguiding system for comparison with the results of the variational technique. Preliminary results for $\beta_0/k$ indicate excellent agreement with the variational calculation in the strong- and weak-coupling regions ($kb \lesssim 0.5$ and $kb \gtrsim 1.5$) and our preliminary results indicate a discrepancy of less than $\sim 6\%$. The numerical results in the intermediate regime are qualitatively very similar to the results of the variational calculation, but differ more significantly in their numerical values. An anomaly of the form seen here in figures 5 and 6 is also present in the numerical work. As expected from the variational procedure the numerical result always lies above the analytic variational result. A complete analysis and detailed comparison with this work is in preparation, but as yet incomplete.

## 6. Estimation of the field profile of the lowest-order mode

The trial partial refractive index inhomogeneity function used in the variational procedure described in section 3 will now be used to calculate approximate expressions for the lowest-order mode field profile of the parallel coupled waveguide system. The method used to do this follows that used by Constantinou and Jones [12], in which an approximate closed form expression for the propagator of the waveguide system is obtained.

It is also possible to derive the results obtained in section 3 using a local trial optical action [13]

$$\mathcal{A}_{\Omega}^{x_0} = \int_0^{\mu} \mathrm{d}\zeta \left[\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\Omega^2(x_0)\big(x - x_0\big)^2\right] \tag{65}$$

which is that of a system with a quadratic partial refractive index inhomogeneity function with a curvature $\Omega\,(x_0)$ centred around some point $x_0$ (using units where $k = 1$). This trial optical action will be the starting point for the calculation of an approximate expression for the propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$.

For the case where the partial refractive index inhomogeneity function $n_x'(x, z)$ is independent of $z$, an imaginary propagation distance can be introduced as before via $z_b - z_a = -\mathrm{i}\mu$ so that the propagator given by equation (15) becomes

$$\mathcal{K}_x(x_b, -\mathrm{i}\mu; x_a, 0) = \sum_{n=0}^{\infty} \varphi_n^x(x_b)\,\varphi_n^{x*}(x_a)\ \exp\left\{-\mu\beta_n^x\right\} \tag{66}$$

which latter is analogous to an expression for the density matrix in statistical mechanics [16]. From this expression the lowest-order mode propagation constant $\beta_0^x$ can be obtained from an equation which is analogous to the well known Feynman–Kac formula in quantum mechanics [17]. It is also possible to obtain the field profile of the lowest-order mode from the expansion of the propagator given by equation (66) by taking the limit $\mu \to +\infty$ [16]:

$$\lim_{\mu\to+\infty} \mathcal{K}_x(x_b, -\mathrm{i}\mu; x_a, 0) = \varphi_0^x(x_b)\,\varphi_0^{x*}(x_a)\ \exp\left\{-\mu\beta_0^x\right\}. \tag{67}$$

The partial refractive index inhomogeneity function associated with the optical action of equation (65) (the local trial partial refractive index inhomogeneity function) is $\frac{1}{2}\Omega^2(x_0)\,(x - x_0)^2$. The local trial propagator is now defined as the propagator associated with the local trial partial refractive index inhomogeneity function and is

$$\mathcal{K}_{x\,\Omega}^{x_0}(x_b, z_b; x_a, z_a) = \int_{(x_a,z_a)}^{(x_b,z_b)} \mathcal{D}x \exp\left\{\mathrm{i}\mathcal{A}_{\Omega}^{x_0}\right\} \tag{68}$$

where

$$\mathcal{A}_{\Omega}^{x_0} = \int_{z_a}^{z_b} \left[\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\Omega^2(x_0)\,(x - x_0)^2\right] \mathrm{d}z. \tag{69}$$

The required propagator is the propagator associated with the partial refractive index inhomogeneity function $V(x)$ given by

$$\mathcal{K}_x(x_b, z_b; x_a, z_a) = \int_{(x_a,z_a)}^{(x_b,z_b)} \mathcal{D}x \exp\left\{\mathrm{i}\mathcal{A}\right\} \tag{70}$$

where

$$\mathcal{A} = \int_{z_a}^{z_b} \left[\tfrac{1}{2}\dot{x}^2 - V(x)\right] \mathrm{d}z \tag{71}$$

which can be written as [9, 13]

$$\mathcal{K}_x = \mathcal{K}_{x\,\Omega}^{x_0} \left\langle \exp\left\{\mathrm{i}\left[\mathcal{A} - \mathcal{A}_{\Omega}^{x_0}\right]\right\}\right\rangle_{\mathcal{A}_{\Omega}^{x_0}} \tag{72}$$

using the trial propagator given by equation (68). The averaging is defined by

$$\langle O\rangle_{\mathcal{A}_{\Omega}^{x_0}} = \frac{\int_{(x_a,z_a)}^{(x_b,z_b)} \mathcal{D}x\ O\ \exp\left\{\mathrm{i}\mathcal{A}_{\Omega}^{x_0}\right\}}{\int_{(x_a,z_a)}^{(x_b,z_b)} \mathcal{D}x \exp\left\{\mathrm{i}\mathcal{A}_{\Omega}^{x_0}\right\}} \tag{73}$$

and can be thought of as an average over paths $x$ of the functional $O$ with a weight functional $\exp\left\{\mathrm{i}\mathcal{A}_{\Omega}^{x_0}\right\}$, and is defined in such a way that $\langle 1\rangle_{\mathcal{A}_{\Omega}^{x_0}} = 1$.

In a variational calculation, the expectation value on the right-hand side of equation (72) can be approximated by its first cumulant as [13, 18]

$$\left\langle \exp\left\{\mathrm{i}\left[\mathcal{A}-\mathcal{A}_{\Omega}^{x_0}\right]\right\}\right\rangle_{\mathcal{A}_{\Omega}^{x_0}} \approx \exp\left\{\mathrm{i}\left\langle\mathcal{A}-\mathcal{A}_{\Omega}^{x_0}\right\rangle_{\mathcal{A}_{\Omega}^{x_0}}\right\} \tag{74}$$

but the approximation on the right-hand side of equation (74) is not necessarily smaller than the left-hand side because the measure of integration in the average defined by equation (73) is no longer positive definite [13]. This means that the propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$ given by equation (70) can be approximated by

$$\mathcal{K}_x \approx \mathcal{K}_{x\,\Omega}^{x_0} \exp\left\{\mathrm{i}\left\langle\mathcal{A}-\mathcal{A}_{\Omega}^{x_0}\right\rangle_{\mathcal{A}_{\Omega}^{x_0}}\right\}. \tag{75}$$

The path integral in the definition of the trial propagator $\mathcal{K}_{x\,\Omega}^{x_0}(x_b, z_b; x_a, z_a)$ in equation (68) can be easily evaluated to give

$$\begin{aligned}\mathcal{K}_{x\,\Omega}^{x_0}(x_b, z_b; x_a, z_a) = &\sqrt{\frac{\Omega(x_0)}{2\pi\mathrm{i}\sin\left[\Omega\left(x_0\right)\left(z_b-z_a\right)\right]}} \exp\left\{\frac{\mathrm{i}\Omega(x_0)}{2\sin\left[\Omega(x_0)\left(z_b-z_a\right)\right]}\right.\\ &\times\left[\left(\left(x_a-x_0\right)^2+\left(x_b-x_0\right)^2\right)\cos\left[\Omega\left(x_0\right)\left(z_b-z_a\right)\right]\right.\\ &\left.\left.-2\left(x_a-x_0\right)\left(x_b-x_0\right)\right]\right\}.\end{aligned} \tag{76}$$

The expectation value on the right-hand side of equation (75) is

$$\left\langle\mathcal{A}-\mathcal{A}_{\Omega}^{x_0}\right\rangle_{\mathcal{A}_{\Omega}^{x_0}} = \left\langle\int_{z_a}^{z_b}\left[\tfrac{1}{2}\Omega^2(x_0)\left(x-x_0\right)^2 - V(x)\right]\mathrm{d}z\right\rangle_{\mathcal{A}_{\Omega}^{x_0}}. \tag{77}$$

If the partial refractive index inhomogeneity function $V(x)$ is of the form

$$V(x) = \sum_{n=0}^{N} V_n x^n \tag{78}$$

as is the case for the partial refractive index inhomogeneity function given by equation (37), it may be re-expanded around the point $x_0$ as [13]

$$V(x) = \sum_{n=0}^{N} V_n \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, x_0^{n-k}\left(x-x_0\right)^k. \tag{79}$$

By doing this, the expectation value given by equation (77) can be written as

$$\left\langle\mathcal{A}-\mathcal{A}_{\Omega}^{x_0}\right\rangle_{\mathcal{A}_{\Omega}^{x_0}} = \int_{z_a}^{z_b}\left[\frac{1}{2}\Omega^2(x_0)\left\langle y^2\right\rangle_{\tilde{\mathcal{A}}_{\Omega}^{x_0}} - \sum_{n=0}^{N} V_n \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\, x_0^{n-k}\left\langle y^k\right\rangle_{\tilde{\mathcal{A}}_{\Omega}^{x_0}}\right]\mathrm{d}z \tag{80}$$

where the $y$-variable is defined by $y = x - x_0$, and the averaging on the right-hand side is defined by

$$\langle O\rangle_{\tilde{\mathcal{A}}_{\Omega}^{x_0}} = \frac{\int_{(y_a,z_a)}^{(y_b,z_b)}\mathcal{D}y\; O\;\exp\left\{\mathrm{i}\int_{z_a}^{z_b}\left[\frac{1}{2}\dot{y}^2-\frac{1}{2}\Omega^2(x_0)\,y^2\right]\mathrm{d}z\right\}}{\int_{(y_a,z_a)}^{(y_b,z_b)}\mathcal{D}y\,\exp\left\{\mathrm{i}\int_{z_a}^{z_b}\left[\frac{1}{2}\dot{y}^2-\frac{1}{2}\Omega^2(x_0)\,y^2\right]\mathrm{d}z\right\}} \tag{81}$$

where $y_a = x_a - x_0$ and $y_b = x_b - x_0$. The expectation values on the right-hand side of equation (80), $\left\langle y^k\right\rangle_{\tilde{\mathcal{A}}_{\Omega}^{x_0}}$, can be calculated using the generating functional $\Phi$ given by [12, 13]

$$\Phi \equiv \left\langle\exp\left\{\mathrm{i}\int_{z_a}^{z_b}\mathrm{d}z\, f(z)\, y(z)\right\}\right\rangle_{\tilde{\mathcal{A}}_{\Omega}^{x_0}}. \tag{82}$$

The path integrals in the definition of the characteristic functional can be evaluated [9] and we find that

$$\Phi = \exp\left\{\frac{\mathrm{i}}{\sin[\Omega(x_0)(z_b - z_a)]}\left[(x_b - x_0)\int_{z_a}^{z_b} f(z)\sin[\Omega(x_0)(z - z_a)]\,\mathrm{d}z\right.\right.$$
$$+ (x_a - x_0)\int_{z_a}^{z_b} f(z)\sin[\Omega(x_0)(z_b - z)]\,\mathrm{d}z$$
$$\left.\left.+\frac{1}{2\Omega(x_0)}\int_{z_a}^{z_b}\int_{z_a}^{z_b} g(z,u)f(z)f(u)\,\mathrm{d}u\,\mathrm{d}z\right]\right\} \tag{83}$$

where

$$g(z,u) = \begin{cases} -\sin[\Omega(x_0)(u - z_a)]\sin[\Omega(x_0)(z_b - z)] & \text{for } z > u \\ -\sin[\Omega(x_0)(z_b - u)]\sin[\Omega(x_0)(z - z_a)] & \text{for } z < u. \end{cases} \tag{84}$$

Using this generating functional the $n$-point correlation function can be obtained by carrying out $n$ functional derivatives with respect to the function $f(z)$ [13]:

$$\langle y(z_1)\,y(z_2)\ldots y(z_n)\rangle_{\tilde{\mathcal{A}}_\Omega^{x_0}} = \frac{1}{\mathrm{i}^n}\frac{\delta^n}{\delta f(z_1)\,\delta f(z_2)\ldots\delta f(z_n)}\Phi\bigg|_{f(z)=0}. \tag{85}$$

For the case of the partial refractive index inhomogeneity function $V(x)$ given by equation (37), the expectation value in equation (80) becomes

$$\left\langle\mathcal{A} - \mathcal{A}_\Omega^{x_0}\right\rangle_{\mathcal{A}_\Omega^{x_0}} = \int_{z_a}^{z_b}\left[\tfrac{1}{2}\Omega^2(x_0)\left\langle y^2(z)\right\rangle_{\tilde{\mathcal{A}}_\Omega^{x_0}} - a^4b^4 - a^4x_0^2\left(x_0^2 - 2b^2\right)\right.$$
$$-4a^4x_0\left(x_0^2 - b^2\right)\langle y(z)\rangle_{\tilde{\mathcal{A}}_\Omega^{x_0}} - 2a^4\left(3x_0^2 - b^2\right)\left\langle y^2(z)\right\rangle_{\tilde{\mathcal{A}}_\Omega^{x_0}}$$
$$\left.-4a^4x_0\left\langle y^3(z)\right\rangle_{\tilde{\mathcal{A}}_\Omega^{x_0}} - a^4\left\langle y^4(z)\right\rangle_{\tilde{\mathcal{A}}_\Omega^{x_0}}\right]\mathrm{d}z. \tag{86}$$

Each of the expectation values on the right-hand side of equation (86) can be calculated using the generating functional given by equation (82) as described above, and the integration with respect to $z$ can be easily carried out. An approximate expression for the propagator $\mathcal{K}_x(x_b, z_b; x_a, z_a)$ is therefore obtained by combining the resulting expression for $\left\langle\mathcal{A} - \mathcal{A}_\Omega^{x_0}\right\rangle_{\mathcal{A}_\Omega^{x_0}}$ with the expression for the local trial propagator $\mathcal{K}_{x\,\Omega}^{x_0}(x_b, z_b; x_a, z_a)$ given by equation (76). By writing the approximate expression for the propagator in terms of $-\mathrm{i}\mu = z_b - z_a$ and then using the expression in equation (67) we obtain approximate expressions for the lowest-order mode profile and partial propagation constant given by

$$\varphi_0^x(x) \approx \left(\frac{\Omega}{\pi}\right)^{1/4}\exp\left\{-\frac{a^4b^2}{2\Omega^2} - \frac{1}{8} + \frac{9a^4}{16\Omega^3} + \frac{3a^4x_0^2}{2\Omega^2}\right\}$$
$$\times\exp\left\{\frac{4a^4x_0}{\Omega}\left[x_0^2 - b^2 - \frac{1}{\Omega}\right](x - x_0)\right.$$
$$+\left[\frac{a^4b^2}{\Omega} - \frac{\Omega}{4} - \frac{3a^4}{4\Omega^2} - \frac{3a^4x_0^2}{\Omega}\right](x - x_0)^2$$
$$\left.-\frac{4a^4x_0}{3\Omega}(x - x_0)^3 - \frac{a^4}{4\Omega}(x - x_0)^4\right\} \tag{87}$$

and

$$\beta_0^x \approx a^4\left(b^4 + \frac{3}{4\Omega^2} - \frac{b^2}{\Omega} + x_0^2\left[x_0^2 - 2b^2 + \frac{3}{\Omega}\right]\right) + \frac{\Omega}{4} \tag{88}$$

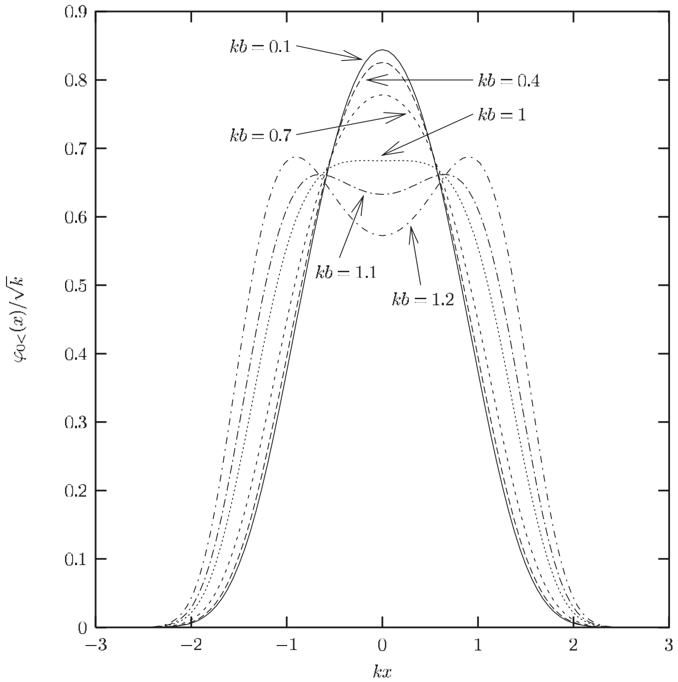


**Figure 7.** The (unnormalized) dimensionless approximate lowest-order mode field profile $\varphi_{0<}(x)/\sqrt{k}$ given by equation (91) as a function of the dimensionless position $kx$ for various values of the dimensionless guide separation $kb$ with $a = k$.

respectively. It has been found that the values of $\Omega$ and $x_0$ which provide the best bound for the lowest-order propagation constant depend on the value of the parameter $b$ with

$$\Omega = \Omega_<^* \qquad \text{and} \qquad x_0 = 0 \qquad\qquad \text{for} \quad b < \tilde{b}_c \tag{89}$$

$$\Omega = \Omega_>^* \qquad \text{and} \qquad x_0 = \pm\sqrt{b^2 - \frac{3}{2\Omega_>^*}} \qquad \text{for} \quad b > \tilde{b}_c. \tag{90}$$

Equation (88) provides the same approximation for the propagation constant of the lowest-order mode as equation (60) for the cases given by equations (89) and (90).

For values of $b < \tilde{b}_c$, equation (87) provides the approximation

$$\varphi_{0<}(x) \approx \left(\frac{k\Omega_<^*}{\pi}\right)^{1/4} \exp\left\{-\frac{a^4b^2}{2\Omega_<^{*2}} - \frac{1}{8} + \frac{9a^4}{16k\Omega_<^{*3}}\right\} \times \exp\left\{\left[\frac{a^4b^2k}{\Omega_<^*} - \frac{k\Omega_<^*}{4} - \frac{3a^4}{4\Omega_<^{*2}}\right]x^2 - \frac{a^4k}{4\Omega_<^*}x^4\right\} \tag{91}$$

for the field profile of the lowest-order mode. The field profile given by equation (91) is plotted in figure 7 as a function of the dimensionless position $kx$ for a number of values of the dimensionless guide separation $kb$.

For values of $b > \tilde{b}_c$, equation (87) provides the approximation

$$\varphi_{>\pm}(x) \approx \left(\frac{k\Omega_>^*}{\pi}\right)^{1/4} \exp\left\{-\frac{33a^4}{16k\Omega_>^{*3}} - \frac{37a^4b^2}{4\Omega_>^{*2}} + \frac{7a^4b^4k}{\Omega_>^*} + \frac{1}{4} - \frac{b^2k\Omega_>^*}{4}\right\} \times \exp\left\{\mp\left[\frac{10a^4}{\Omega_>^{*2}} + \frac{a^4b^2k}{\Omega_>^*} - \frac{k\Omega_>^*}{2}\right]\left(b^2 - \frac{3}{2k\Omega_>^*}\right)^{1/2} x\right.$$

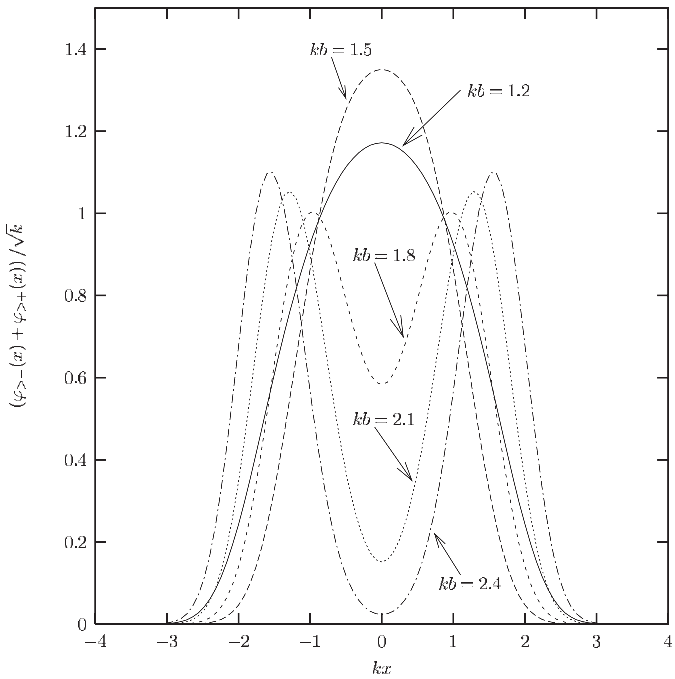


**Figure 8.** The (unnormalized) symmetric combination $(\varphi_{>-}(x) + \varphi_{>+}(x))/\sqrt{k}$ of the (normalized) dimensionless approximate degenerate mode field profiles $\varphi_{>+}(x)/\sqrt{k}$ and $\varphi_{>-}(x)/\sqrt{k}$ given by equation (92) as a function of the dimensionless position $kx$ for various values of the dimensionless guide separation $kb$ with $a = k$.

$$+\left[\frac{a^4b^2k}{2\Omega_>^*} - \frac{k\Omega_>^*}{4}\right]x^2 \mp \frac{a^4k}{3\Omega_>^*}\left(b^2 - \frac{3}{2k\Omega_>^*}\right)^{1/2}x^3 - \frac{a^4k}{4\Omega_>^*}x^4\Bigg\} \tag{92}$$

which are the degenerate field profiles corresponding to the minima in the $\mu \to +\infty$ limit of the approximate effective refractive index shown in figure 4. These minima occur at

$$x = \pm\sqrt{b^2 - \frac{3}{2k\Omega_>^*}}. \tag{93}$$

The field profile of the lowest-order mode in this case is given by the symmetric combination of $\varphi_{>-}(x)$ and $\varphi_{>+}(x)$:

$$\varphi_{0>}(x) = \frac{1}{N}\{\varphi_{>-}(x) + \varphi_{>+}(x)\} \tag{94}$$

where $N$ is a normalization factor. The symmetric combination of the normalized versions of the degenerate field profiles $\varphi_{>-}(x)$ and $\varphi_{>+}(x)$ is plotted in figure 8 as a function of the dimensionless position $kx$ for a number of values of the dimensionless guide separation $kb$.

Figure 7 shows the behaviour of the lowest-order mode field profile in the strong-coupling limit and is in qualitative agreement with the results of the previous variational calculation by Constantinou and Jones. Figure 8 shows the behaviour in the weak-coupling limit, which is a new result arising from the application of a variational procedure to the parallel coupled waveguide system. Figures 7 and 8 display the behaviour of the lowest-order mode field profile which is physically expected. In the strong-coupling limit, when the separation of the guides is small compared with their width, the field profile is peaked at $x = 0$ between the two guides, as shown in figure 7. As the coupling between the guides becomes weaker, the field profile is peaked at positions close to the centres of each individual guide. These profiles are consistent with the comments made at the end of section 5. The fact that the maxima of the field profile do not occur exactly at the centres of the two guides is due to a small overlap between the guides, indicating that the starting point for the coupled mode theory is not valid [2, 8].

It has been pointed out in [12] that the model refractive index shown in figure 3 can be used to model the refractive index of the parallel waveguide system provided that the mode fields in the regions where the model refractive index is negative (and therefore unphysical) are negligible. This assumption must be checked *a posteriori*, and it can be seen to be true from the field profiles shown in figures 7 and 8.

Preliminary data from the numerical work referred to in section 5 indicate close agreement with the field profile of the lowest-order mode in the strong-coupling limit although there are some qualitative differences apparent in the weak-coupling results, which we ascribe to the relative insensitivity of the associated propagation constant to the choice of the variational trial function. In the physically relevant region within and between the two guides, the discrepancy between the result shown in figure 7 for $kb = 0.1$ and the corresponding numerical result appears to be no greater than $\sim 0.4\%$. In general, estimation of the field profiles of the higher-order modes is a difficult problem using the variational method and requires a major extension of this work. Some numerical data on the first-order mode are in preparation and will be published in the future.

## 7. Estimation of the propagation constants of the higher-order modes of the system and the beat length

As mentioned in section 1, the close proximity of the two parallel waveguides in the system gives rise to a periodic variation in intensity along each guide. The distance over which a period of this variation occurs is the beat length of the system, denoted by $\Delta z_{\mathrm{beat}}$ using the coordinate axes shown in figure 2. The beat length $\Delta z_{\mathrm{beat}}$ is given by [8, 12]

$$\Delta z_{\mathrm{beat}} = \frac{2\pi}{\beta_0 - \beta_1} \tag{95}$$

where $\beta_0$ is the propagation constant of the lowest-order mode and $\beta_1$ is the propagation constant of the first-order mode. This means that it is necessary to calculate (approximate) expressions for the propagation constants of the higher-order modes in addition to the lowest-order mode.

A calculation of the propagation constant of the first-order mode was carried out using a variational technique by Constantinou and Jones [12]. They found that the difference between the propagation constants of the lowest-order mode and the first-order mode, $\Delta\beta = \beta_0 - \beta_1$, was determined by the trial curvature $c$ given by equation (64). However, Feynman and Kleinert [14] have shown that for the case of the quantum mechanical double-well potential the trial frequency in their variational procedure only approximates the difference between the first excited-state energy and the ground-state energy for very strong coupling between the two wells of the potential (i.e. when the separation of the two wells is very small). This is due to the fact that the approximation described in section 3 lacks the ability to describe tunnelling when the waveguides are weakly coupled [14]. It is clear from this that another method of obtaining the quantity $\Delta\beta$ is required to calculate the beat length when the waveguides are more weakly coupled.

Kleinert [13] describes a method of calculating approximate expressions for the excited energy levels of a quantum mechanical system based on the Feynman–Kleinert variational procedure that was described in section 3. In this section Kleinert's method will be formulated in terms of the quantities of the optical problem and then applied to the parallel coupled waveguide system in order to calculate the propagation constants of the higher-order modes of the system.

The method described by Kleinert [13] to calculate the excited energy levels consists of calculating an optimized expectation value of the propagator between the mode field profiles of the system described by the trial propagator given by equation (68). This method is expected to provide a good approximation to the excited energy levels of a quantum mechanical system if the potential describing the system is similar to the harmonic oscillator potential [13].

The first step in Kleinert's method is to form the projections

$$Z_n(x_0) \equiv \int \mathrm{d}x_b\,\mathrm{d}x_a\,\psi^{x*}_{\Omega n}(x_b - x_0)\,\mathcal{K}_x(x_b, -\mathrm{i}\mu; x_a, 0)\;\psi^{x}_{\Omega n}(x_a - x_0) \qquad (96)$$

where $\mathcal{K}_x(x_b, z_b - z_a; x_a, 0)$ is the propagator of the waveguiding system of interest and $\psi^{x}_{\Omega n}(x - x_0)$ is the partial field profile of the $n$th-order mode of the waveguiding system with the trial partial refractive index inhomogeneity function. The imaginary propagation distance propagator, $\mathcal{K}_x(x_b, -\mathrm{i}\mu; x_a, 0)$, is given by

$$\mathcal{K}_x(x_b, -\mathrm{i}\mu; x_a, 0) = \int_{(x_a,0)}^{(x_b,\mu)} \mathcal{D}x\,\mathrm{e}^{-\mathcal{A}} \qquad (97)$$

where $\mathcal{A}$ is the optical Euclidean action for the waveguiding system given by

$$\mathcal{A}[x] = \int_0^{\mu} \mathrm{d}\zeta\,\left[\tfrac{1}{2}\dot{x}^2 + V(x)\right] \qquad (98)$$

so that equation (96) can be written as

$$Z_n(x_0) = \int \mathrm{d}x_b\,\mathrm{d}x_a\,\psi^{x*}_{\Omega n}(x_b - x_0)\int_{(x_a,0)}^{(x_b,\mu)} \mathcal{D}x\,\mathrm{e}^{-\mathcal{A}}\,\psi^{x}_{\Omega n}(x_a - x_0)\,. \qquad (99)$$

In a similar way to that described in section 5, the expression for $Z_n$ in equation (99) can be written in terms of an average as

$$Z_n = \left(\int \mathrm{d}x_b\,\mathrm{d}x_a\,\psi^{x*}_{\Omega n}(x_b - x_0)\int_{(x_a,0)}^{(x_b,\mu)} \mathcal{D}x\,\mathrm{e}^{-\mathcal{A}^{x_0}_{\Omega}}\,\psi^{x}_{\Omega n}(x_a - x_0)\right)\left\langle \mathrm{e}^{-(\mathcal{A}-\mathcal{A}^{x_0}_{\Omega})}\right\rangle_{\Omega n} \qquad (100)$$

where the expectation value is defined by

$$\langle O\rangle_{\Omega n} \equiv \frac{\int \mathrm{d}x_b\,\mathrm{d}x_a\,\psi^{x*}_{\Omega n}(x_b - x_0)\int_{(x_a,0)}^{(x_b,\mu)} \mathcal{D}x\;O\,\mathrm{e}^{-\mathcal{A}^{x_0}_{\Omega}}\,\psi^{x}_{\Omega n}(x_a - x_0)}{\int \mathrm{d}x_b\,\mathrm{d}x_a\,\psi^{x*}_{\Omega n}(x_b - x_0)\int_{(x_a,0)}^{(x_b,\mu)} \mathcal{D}x\,\mathrm{e}^{-\mathcal{A}^{x_0}_{\Omega}}\,\psi^{x}_{\Omega n}(x_a - x_0)} \qquad (101)$$

and $\mathcal{A}^{x_0}_{\Omega}$ is the optical Euclidean action associated with the trial partial refractive index inhomogeneity function and is given by

$$\mathcal{A}^{x_0}_{\Omega} = \int_0^{\mu} \mathrm{d}\zeta\left[\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\Omega^2\left(x - x_0\right)^2\right]. \qquad (102)$$

The term multiplying the expectation value on the right-hand side of equation (100) is the contribution of the $n$th-order mode of the waveguiding system with the trial partial refractive index inhomogeneity function to the trial optical partition function [13] and can be shown to be equal to $\exp\{-\mu\,\Omega(n + \frac{1}{2})\}$, so that equation (100) becomes

$$Z_n = \exp\left\{-\mu\,\Omega\left(n + \tfrac{1}{2}\right)\right\}\left\langle \exp\left\{-\left(\mathcal{A} - \mathcal{A}^{x_0}_{\Omega}\right)\right\}\right\rangle_{\Omega n}. \qquad (103)$$

The expectation value on the right-hand side of equation (103) can be approximated in the same way as described in section 5, and equation (103) then becomes

$$Z_n \approx \exp\left\{-\mu\,\Omega\left(n + \tfrac{1}{2}\right)\right\}\,\exp\left\{-\left\langle \mathcal{A} - \mathcal{A}^{x_0}_{\Omega}\right\rangle_{\Omega n}\right\}. \qquad (104)$$

We now assume once more that the partial refractive index inhomogeneity function of interest $V(x)$ is of the form

$$V(x) = \sum_{k=0}^{N} V_k\, x^{2k} \tag{105}$$

so that the difference in optical Euclidean actions $\mathcal{A} - \mathcal{A}_\Omega^{x_0}$ can be written as

$$\mathcal{A} - \mathcal{A}_\Omega^{x_0} = \int_0^\mu \mathrm{d}\zeta \left[ \sum_{k=0}^{N} V_k\, x^{2k} - \tfrac{1}{2}\Omega^2\, (x - x_0)^2 \right]. \tag{106}$$

As described by Kleinert [13], the calculation of the expectation value of $\mathcal{A} - \mathcal{A}_\Omega^{x_0}$ is simplified if the optimal value of $x_0$ (determined in the way described in section 5) is at the origin. This was found in section 5 to be true for the case where the waveguides are strongly coupled. For this case the expression in equation (106) becomes

$$\mathcal{A} - \mathcal{A}_\Omega^{x_0=0} = \int_0^\mu \mathrm{d}\zeta \left[ \sum_{k=0}^{N} V_k\, x^{2k} - \tfrac{1}{2}\Omega^2 x^2 \right]. \tag{107}$$

The expectation value of $\mathcal{A} - \mathcal{A}_\Omega^{x_0=0}$ is given by

$$\left\langle \mathcal{A} - \mathcal{A}_\Omega^{x_0=0} \right\rangle_{\Omega\, n} = \int_0^\mu \mathrm{d}\zeta \left[ \sum_{k=0}^{N} V_k \left\langle x^{2k} \right\rangle_{\Omega\, n} - \tfrac{1}{2}\Omega^2 \left\langle x^2 \right\rangle_{\Omega\, n} \right] \tag{108}$$

with the expectation values on the right-hand side, $\left\langle x^{2k} \right\rangle_{\Omega\, n}$, being given by [13]

$$\left\langle x^{2k} \right\rangle_{\Omega\, n} = \frac{1}{\Omega^k} n_{2k} \tag{109}$$

where

$$\begin{aligned} n_2 &= n + \tfrac{1}{2} \\ n_4 &= \tfrac{3}{2}\left(n^2 + n + \tfrac{1}{2}\right) \\ n_6 &= \tfrac{5}{4}\left(2n^3 + 3n^2 + 4n + \tfrac{3}{2}\right) \\ n_8 &= \tfrac{1}{16}\left(70n^4 + 140n^3 + 344n^2 + 280n + 105\right) \end{aligned}$$

etc and $n_0 = 1$ (so that $\langle 1 \rangle_{\Omega\, n} = 1$), where $n$ is an integer labelling the mode of the trial waveguiding system used in the expectation value in equation (109) ($n$ can take values $0, 1, 2, \ldots$). Thus equation (108) becomes

$$\left\langle \mathcal{A} - \mathcal{A}_\Omega^{x_0=0} \right\rangle_{\Omega\, n} = \left[ \sum_{k=0}^{N} V_k \frac{1}{\Omega^k} n_{2k} - \frac{1}{2}\Omega\, n_2 \right] \mu. \tag{110}$$

The resulting expression for $Z_n$ is therefore

$$Z_n \approx \exp\left\{ -\mu \left[ \frac{1}{2}\Omega\, n_2 + \sum_{k=0}^{N} V_k \frac{1}{\Omega^k} n_{2k} \right] \right\}. \tag{111}$$

By analogy with Kleinert [13], we now define a quantity $W_n$ defined by $Z_n = \exp\{-\mu W_n\}$, so that $W_n$ is given by

$$W_n = \frac{1}{2}\Omega\, n_2 + \sum_{k=0}^{N} V_k \frac{1}{\Omega^k} n_{2k}. \tag{112}$$

In the limit $\mu \to +\infty$, a minimization of $W_n$ with respect to the parameters $x_0$ and $\Omega \equiv \Omega(x_0)$ yields an approximation to the partial propagation constant $\beta_n^x$ of the $n$th-order mode of the waveguiding system [13]. This approximation to $\beta_n^x$ will be denoted by $\beta_n^{x\,(1)}$. The required optimization is carried out by finding the value of the trial curvature $\Omega$ which minimizes the $W_n$.

This method can now be applied to the waveguiding system of interest which is the parallel coupled waveguide system and has a partial refractive index inhomogeneity function given by equation (37). For this system, $W_n$ of equation (112) becomes

$$W_n = a^4b^4 - \frac{2a^4b^2}{\Omega}n_2 + \frac{\Omega}{2}n_2 + \frac{a^4}{\Omega^2}n_4. \tag{113}$$

Minimizing this expression with respect to $\Omega$ shows that the minimum value of $W_n$, which approximates the partial propagation constant of the $n$th-order mode of the parallel coupled waveguide system, occurs when $\Omega$ satisfies the equation

$$n_2\Omega^3 + 4a^4b^2n_2\Omega - 4a^4n_4 = 0. \tag{114}$$

The real root of this equation is denoted by $\tilde{\Omega}_n$ and given by

$$\tilde{\Omega}_n = \left(\frac{2a^4n_4}{n_2}\right)^{1/3}\left\{\left[\left(1+\left(\frac{b}{b_n}\right)^6\right)^{1/2}+1\right]^{1/3} - \left[\left(1+\left(\frac{b}{b_n}\right)^6\right)^{1/2}-1\right]^{1/3}\right\} \tag{115}$$

where

$$b_n = \left(\frac{27n_4^2}{16a^4n_2^2}\right)^{1/6}. \tag{116}$$

This means that the approximation for the partial propagation constant of the $n$th-order mode is given by

$$\beta_n^{x\,(1)} = a^4b^4 - \frac{2a^4b^2}{\tilde{\Omega}_n}n_2 + \frac{\tilde{\Omega}_n}{2}n_2 + \frac{a^4}{\tilde{\Omega}_n^2}n_4. \tag{117}$$

Using equations (117) and (16), the approximation to the propagation constant of the first-order mode of the parallel coupled waveguide system is given by

$$\beta_1 \approx k - \frac{3\tilde{\Omega}_1}{4} + \frac{3a^4b^2}{\tilde{\Omega}_1} - ka^4b^4 - \frac{15a^4}{4k\tilde{\Omega}_1^2} \tag{118}$$

where

$$\tilde{\Omega}_1 = \left(\frac{5a^4}{k}\right)^{1/3}\left\{\left[\left(1+\frac{64}{675}k^2a^4b^6\right)^{1/2}+1\right]^{1/3} - \left[\left(1+\frac{64}{675}k^2a^4b^6\right)^{1/2}-1\right]^{1/3}\right\}. \tag{119}$$

This approximation for the propagation constant of the first-order mode is only valid when the waveguides are strongly coupled since the optimal value of $x_0$ used in this calculation was $x_0 = 0$. Using equation (117) it is also possible to obtain an approximate expression for the lowest-order mode propagation constant. This approximate expression is the same as that obtained in section 5 for strongly coupled waveguides and given by equation (60) with $b < \tilde{b}_c$. The approximation for the beat length of the parallel coupled waveguide system is obtained by inserting the expressions given by equations (60) and (118) into (95). The beat length obtained in this way is plotted in figure 9 as a function of the dimensionless guide separation $kb$. The result obtained by Constantinou and Jones [12] for the propagation constant of the first-order mode is

$$\beta_1^{\mathrm{C-J}} \approx k + \left[\frac{a^4b^2}{c^2} - \frac{5}{4}\right]c - ka^4b^4 - \frac{3a^4}{4kc^2} \tag{120}$$

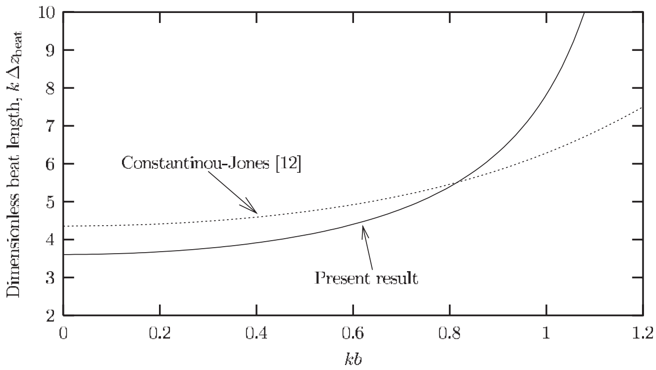


**Figure 9.** The dimensionless approximate beat length $k\,\Delta z_{\mathrm{beat}}$ of the parallel coupled waveguide system as a function of the dimensionless guide separation $kb$ with $a = k$, compared with the result obtained by Constantinou and Jones [12] and given by equation (121).

where $c$ is given by equation (64). From equations (63) and (120), it can be seen that the approximation obtained by Constantinou and Jones for the beat length of the parallel coupled waveguide system is

$$\Delta z_{\mathrm{beat}}^{\mathrm{C-J}} = \frac{2\pi}{\beta_0^{\mathrm{C-J}} - \beta_1^{\mathrm{C-J}}} \approx \frac{2\pi}{c} \tag{121}$$

with $c$ given by equation (64). The expression in equation (121) is also plotted in figure 9 as a function of the dimensionless guide separation $kb$.

Figure 9 shows that the new approximate beat length increases faster with increasing guide separation than does the previous result of Constantinou and Jones. Other analytical calculations using the coupled mode theory obtain expressions for the beat length which increase at least exponentially with the separation of the waveguides [2]. It is therefore clear that the new approximate beat length is in better agreement with these results than is the approximate beat length obtained by Constantinou and Jones. The calculation of the propagation constant of the first-order mode for the case where the waveguides are weakly coupled (i.e. when the optimal value of $x_0$ does not lie at the origin, as was described in section 5) is more complicated and will not be included here. This makes a detailed comparison with the results of the other analytical calculations not meaningful.

As mentioned at the end of section 5, estimates of the propagation constants of the waveguiding system can be calculated numerically. The beat length calculated from the numerical propagation constants of the lowest-order and first-order modes agrees well with the analytical approximation of the beat length for the case where the waveguides are strongly coupled. The discrepancy between the numerical and analytical results for the beat length in the region $0 < kb < 1.1$ is less than $\sim$3%, indicating the high accuracy of the analytical approximations calculated in this paper.

## 8. Conclusions

The path integral approach to the study of optical propagation presents an exciting way of describing many problems of propagation in guiding systems, in an intuitively appealing way by providing a global picture of propagation.

The new variational procedure is shown to provide a better variational bound for the lowest-order propagation constant than the result of the calculation in [12]. It predicts a definite distinction between the cases of strongly and weakly coupled waveguides and is able to make new predictions about the intermediate regime, which has proved previously inaccessible. The approximate expressions for the lowest-order mode propagation constant obtained agree with the numerical result for this quantity to a very high degree, except in the region where the propagation constant behaves in an unphysical way.

The field profile of the lowest-order mode obtained from the variational method agrees approximately with that obtained from a previous variational calculation in the strong-coupling limit and provides a new result in the weak-coupling limit. These field profiles are found to be more physically appealing than those calculated previously using a variational procedure [12].

The expression for the beat length obtained for the case where the waveguides are strongly coupled is in better agreement with other analytical calculations and the numerical result than is the result of the previous variational calculation by Constantinou and Jones.

The new calculation is also able to make predictions in the previously inaccessible intermediate region between strong and weak coupling. The approximate results calculated in this paper are important because of the need for analytical results which can be used to guide experimental and numerical work on the design of integrated optical devices.

Further work on this problem is required in order to make a detailed comparison of the results of this variational calculation with other analytical and numerical calculations. It would also be useful to make a comparison between the analytical work and a numerical scheme for the evaluation of the path integral for the optical propagator describing the system.

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